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A NEW FORMALISM IN
PERTURBATION THEORY USING
CONTINUED FRACTIONS

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PREFACE

The Project RAND research program consists in part of basic supporting studies in mathematics. One aspect of this involves the investigation of the approximate behavior of solutions of differential equations.

In the present Memorandum the author discusses the problem of determining useful analytic approximations to solutions of equations of the form

$$L(u) + (a(p) + \lambda b(p))u = 0$$

where $L(u)$ is a linear differential operator.

SUMMARY

A problem of continuing interest is that of obtaining approximate solutions of the functional equation

$$L(u) + (a(p) + \lambda b(p))u = 0,$$

where L is a linear transformation, in terms of the solution of the unperturbed equation

$$L(u) + a(p)u = 0.$$

Using the Green's function, or equivalent techniques, and regarding the term involving λ as a forcing term, we can convert the first equation to the form

$$u = f + \lambda T(u),$$

where T is a linear transformation.

We present a new approach to problems of this nature using the classical technique of continued fractions.

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A NEW FORMALISM IN PERTURBATION THEORY USING CONTINUED FRACTIONS

1. INTRODUCTION

A problem of continuing interest is that of obtaining approximate solutions of the functional equation

$$(1.1) \quad L(u) + (a(p) + \lambda b(p))u = 0,$$

where L is a linear differential operator, in terms of the solution of the unperturbed equation

$$(1.2) \quad L(u) + a(p)u = 0.$$

Using the Green's function of (1.2), or equivalent techniques, and regarding the term involving λ as a forcing term, we can convert (1.1) into an equation of the form

$$(1.3) \quad u = f + \lambda T(u),$$

where T is a linear transformation.

We shall present a new approach to problems of this nature using the classical technique of continued fractions. In subsequent papers we shall discuss the rigorous aspects, and the value of this approach in obtaining rational approximations outside the circle of convergence of the Liouville-Neumann solution of (1.3).

2. CONTINUED FRACTION EXPANSION

From (1.3) we have

$$(2.1) \quad T^n(u) = T^n(f) + \lambda T^{n+1}(u), \quad n = 0, 1, 2, \dots$$

Hence, combining the n -th and $(n+1)$ -st expressions,

$$(2.2) \quad T^n(u) = \left(\frac{T^n(f)}{T^{n+1}(f)} + \lambda \right) T^{n+1}(u) - \lambda \frac{T^n(f)}{T^{n+1}(f)} T^{n+2}(u),$$

a three-term recurrence relation. Thus

$$(2.3) \quad \frac{T^n(u)}{T^{n+1}(u)} = \left(\frac{T^n(f)}{T^{n+1}(f)} + \lambda \right) - \frac{\lambda T^n(f)/T^{n+1}(f)}{T^{n+1}(u)/T^{n+2}(u)}.$$

Thus, formally (setting $T^n(f) = T_n$),

$$(2.4) \quad \frac{u}{T(u)} = \left(\frac{T_0}{T_1} + \lambda \right) - \frac{\lambda T_0/T_1}{\left(\frac{T_1}{T_2} + \lambda \right)} - \frac{\lambda T_1/T_2}{\left(\frac{T_2}{T_3} + \lambda \right)} - \dots$$

Returning to (1.3), we have $T(u) = (u - f)/\lambda$. Combining this result with (2.4), we have the following representation for the solution:

$$(2.5) \quad u = \frac{f}{1 - \frac{\lambda}{\left(\frac{T_0}{T_1} + \lambda \right)} - \frac{\lambda T_0/T_1}{\left(\frac{T_1}{T_2} + \lambda \right)} - \dots} = \frac{f}{1 - \frac{\lambda T_1}{(T_0 + \lambda T_1)} - \frac{\lambda T_0 T_2}{(T_1 + \lambda T_2)} - \dots}.$$

3. EXAMPLES—I

If we set

$$T(u) = \int_0^x u dx_1,$$

so that $u = e^{\lambda x}$, we see that $T_n = x^n/n!$,

$T_n/T_{n+1} = (n+1)/x$. Hence, (2.5) yields

$$(3.1) \quad e^{\lambda x} = \frac{1}{1 - \frac{\lambda}{(\frac{1}{x} + \lambda)} - \frac{\lambda/x}{(\frac{2}{x} + \lambda)} - \dots} = \frac{1}{1 - \frac{\lambda x}{1 + \lambda x} - \frac{\lambda x}{2 + \lambda x} - \dots},$$

a classical result.

4. EXAMPLES—II

Consider the case where we replace λ by $-\lambda$,

$f \geq 0$ and T is a positivity-preserving transformation.

Then a formal solution of $u + \lambda T(u) = f$ has the form

$$(4.1) \quad u = \frac{f}{1 + \frac{\lambda}{\left(\frac{T_0}{T_1} - \lambda\right)} + \frac{\lambda T_0/T_1}{\left(\frac{T_1}{T_2} - \lambda\right)} + \dots}.$$

Hence if $T_n/T_{n+1} - \lambda \geq 0$ for all n , we see that all of the convergents to (4.1) are obviously nonnegative.

We thus obtain a simple sequence of lower and upper bounds to the solution of $u + \lambda T(u) = f$.

5. EXAMPLES—III

Interesting analogues of asymptotic expansions are obtained upon considering equations of the form

$$(5.1) \quad u = f + \epsilon \frac{d^n u}{dt^n},$$

and the matrix analogues.